Supersymmetry in de Sitter space

1979 J. Phys. A: Math. Gen. 12 1751

(http://iopscience.iop.org/0305-4470/12/10/021)

J. Phys. A: Math. Gen., Vol. 12, No. 10, 1979. Printed in Great Britain

Supersymmetry in de Sitter space

D Großer

Institut für Theoretische Physik, Universität Tübingen, D-7400 Tübingen, Auf der Morgenstelle 14, West Germany

Received 14 August 1978

Abstract. We investigate the graded Lie algebra of generators of symmetries assuming that the even subspace of this algebra is spanned by the generators of the de Sitter group and of internal symmetries. We find that the generatoɪs of the de Sitter group form the Lie algebra of O(3, 2) and not of O(4, 1) and that, as in the case of conformal symmetry, there is a complete fusion between geometric and internal symmetries. In particular the internal symmetry group SO(N) is generated by Fermi charges.

1. Introduction and results

Extended supergravity theories exhibiting an SO(N) internal symmetry have been constructed recently (Das 1977, Ferrara *et al* 1977, Ferrara and Van Nieuwenhuizen 1976, Freedman 1977, de Wit and Freedman 1977). To incorporate additional interactions one would like to gauge this internal symmetry. This has been done for $N=2$ and $N=3$ by Freedman and Das (1977). However, to preserve local supersymmetry an apparent mass term for the spin-$\frac{3}{2}$ gauge field and a cosmological term must be introduced. It has been pointed out by Deser and Zumino (1977) that because of this cosmological term one has to quantise in a background space with a metric which is a solution of the Einstein equations with a cosmological term and that the simplest and most natural space of this kind is the corresponding de Sitter space. In the resulting quantum field theory the de Sitter algebra replaces the Poincaré algebra. It therefore seems worthwhile to investigate quite generally the structure of the graded Lie algebra of generators of symmetries in the case in which the geometric symmetries are generated by the elements of the de Sitter algebra.

A grading of the de Sitter algebra (O(3, 2)) can be found among the graded Lie algebras given by Freund and Kaplansky (1976): the OSp(4/N). This algebra has also been found by Ferrara (1977) as a sub-algebra of the graded conformal algebra given by Haag, Łopuszański and Sohnius (1975). Nahm (1977) has shown that this algebra and a grading of the Lie algebra of O(4, 1) are essentially the only possibilities for grading the de Sitter algebras. Our analysis differs from that of Nahm. It shows that under the assumptions stated below only the Lie algebra of O(3, 2) admits a grading and it gives the commutation and anticommutation relations in terms of the generators of translations and Lorentz transformations.

We assume that the elements of the graded Lie algebra L are operators in the Hilbert space of physical states and that the even part B of L is spanned by the generators of the de Sitter group P_μ and $M_{\mu\nu}$ and by a finite number of internal symmetry generators B_l. Hence we have

$$[M_{\mu\nu}, M_{\rho\lambda}] = -\mathrm{i}(g_{\mu\rho}M_{\nu\lambda} + g_{\nu\lambda}M_{\mu\rho} - g_{\mu\lambda}M_{\nu\rho} - g_{\nu\rho}M_{\mu\lambda}) \tag{1.1}$$

$$[M_{\mu\nu}, P_\rho] = -\mathrm{i}(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \tag{1.2}$$

$$[P_\mu, P_\nu] = -\mathrm{i}\rho M_{\mu\nu} \tag{1.3}$$

$$[B_l, P_\mu] = [B_l, M_{\mu\nu}] = 0. \tag{1.4}$$

We shall show that $\rho > 0$, i.e. only the Lie algebra of O(3, 2) admits a grading. Finally we assume that there are no Lorentz scalars in the odd part F of L and that an element $G \in F$ implies $G^+ \in F$ and an element $G \in B$ implies $G^+ \in B$. If the S matrix exists then the generators can be characterised by two properties: they commute with the S matrix and they act additively on the states of several incoming particles. In this case the last assumptions can be derived. The first of them follows from the connection between spin and statistics.

We will then prove the following. In the odd part F one can choose a basis Q^L_α ($L = 1, \ldots, \nu$; $\alpha = 1, 2$) and $\bar{Q}^L_{\dot\alpha} = (Q^L_\alpha)^+$ such that the Q^L are two-component spinors:

$$[Q^L_\alpha, M_{\alpha_1\alpha_2}] = -\mathrm{i}(\epsilon_{\alpha\alpha_1}Q^L_{\alpha_2} + \epsilon_{\alpha\alpha_2}Q^L_{\alpha_1}) \tag{1.5}$$

$$[Q^L_\alpha, \bar{M}_{\dot\alpha_1\dot\alpha_2}] = 0 \tag{1.6}$$

where

$$M_{\alpha_1\alpha_2} = -\tfrac{1}{2}\mathrm{i}(\sigma^{\mu\nu}\epsilon)_{\alpha_1\alpha_2}M_{\mu\nu} \tag{1.7}$$

$$\bar{M}_{\dot\alpha_1\dot\alpha_2} = (M_{\alpha_1\alpha_2})^+ \tag{1.8}$$

with $\sigma^{\mu\nu} = \frac{1}{2}\mathrm{i}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$, $(\sigma^\mu)_{\alpha\dot\beta} = (1, \sigma_i)$, $(\bar\sigma^\mu)^{\alpha\beta} = (1, -\sigma_i)$, $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and $\epsilon_{21} = 1$. One can further choose the Q^L_α such that the following algebraic relations hold:

$$\{Q^L_\alpha, \bar{Q}^M_{\dot\beta}\} = \delta^{LM}P_{\alpha\dot\beta} \tag{1.9}$$

$$[Q^L_\alpha, P_{\beta\dot\gamma}] = \rho^{1/2}\epsilon_{\alpha\beta}\bar{Q}^L_{\dot\gamma} \tag{1.10}$$

$$\{Q^L_\alpha, Q^M_\beta\} = \mathrm{i}\rho^{1/2}\delta^{LM}M_{\alpha\beta} + \rho^{1/2}\epsilon_{\alpha\beta}A^{LM} \tag{1.11}$$

where $P_{\alpha\dot\beta} = (\sigma^\mu)_{\alpha\dot\beta}P_\mu$ and the A^{LM} are linear combinations of the internal symmetry generators B_l. From (1.11) one has directly $A^{LM} = -A^{ML}$. Since we will find

$$A^{LM} = (A^{ML})^+ \tag{1.12}$$

the A^{LM} are skew-Hermitian. It will further be shown that

$$[Q^L_\alpha, A^{MN}] = \delta^{LM}Q^N_\alpha - \delta^{LN}Q^M_\alpha \tag{1.13}$$

$$[A^{LM}, A^{L'M'}] = \delta^{LM'}A^{ML'} - \delta^{MM'}A^{LL'} - \delta^{LL'}A^{MM'} + \delta^{ML'}A^{LM'} \tag{1.14}$$

and that the A^{LM} with $M > L$ are linearly independent. Therefore the set of all linear combinations of the A^{LM} with real coefficients is isomorphic to the Lie algebra of SO(ν) and the corresponding representation of SO(ν) is unitary.

Let I be the complex span of the B_l, I_1 the complex span of the A^{LM} and let K be the set of all elements in I which commute with all Q^L_α. Then we will finally show that I_1 and K are invariant subalgebras of the Lie algebra I and that $I = I_1 \oplus K$.

2. The graded Lie algebra of symmetry generators

The proof of the above results is similar to that employed by Haag, Łopuszański and Sohnius (1975). First we use Lorentz invariance to specify a basis in F and to determine the general structure of the commutation and anticommutation relations. Then the coefficients in these algebraic relations are fixed with the help of Jacobi identities.

The odd part F carries a representation of the even part B and hence a representation of the Lorentz algebra. This representation of the Lorentz algebra can be decomposed into irreducibles. The irreducible representations of the Lorentz algebra can be labeled by pairs of indices (j, j'), each integer or half-integer. Suppose now that this decomposition contains the component (j, j'). The carrier space of this component contains an element Q such that

$$[A_3, Q] = jQ \tag{2.1}$$

$$[B_3, Q] = -j'Q \tag{2.2}$$

where $A_i = \frac{1}{2}(J_i + \mathrm{i}K_i)$, $B_i = A_i^+$ and $J_i = (M_{23}, M_{31}, M_{12})$, $K_i = M_{i0}$. Then $\{Q, Q^+\}$ belongs to the even part B and to the irreducible representation $(j+j', j+j')$. B, however, is spanned by P_μ, $M_{\mu\nu}$, and the B_l. Therefore if $j+j' > \frac{1}{2}$ then $\{Q, Q^+\} = 0$. Since $\{Q, Q^+\} = 0$ implies $Q = 0$ we must have $j + j' \leq \frac{1}{2}$. By assumption there are no scalars in F. Hence we can choose a basis in F consisting of ν spinors Q^L_α and ν' conjugate spinors $\bar{Q}^L_{\dot\alpha}$. If Q^L_α is a spinor then $(Q^L_\alpha)^+$ is a conjugate spinor and vice versa. Hence we must have $\nu' = \nu$ and we can choose the $\bar{Q}^L_{\dot\alpha}$ such that $\bar{Q}^L_{\dot\alpha} = (Q^L_\alpha)^+$.

Commuting A_i, B_i with $\{Q^L_\alpha, \bar{Q}^M_{\dot\beta}\}$ one finds that $\{Q^L_\alpha, \bar{Q}^M_{\dot\beta}\}$ transforms under Lorentz transformations in the same way as $P_{\alpha\dot\beta}$. Hence one has

$$\{Q^L_\alpha, \bar{Q}^M_{\dot\beta}\} = c^{LM} P_{\alpha\dot\beta}. \tag{2.3}$$

This equation implies that c is Hermitian. Hence we can choose the Q^L_α such that c is diagonal. Equation (2.3) then implies

$$c^{LL} P_0 = \tfrac{1}{2}\{Q^L_1, (Q^L_1)^+\} + \tfrac{1}{2}\{Q^L_2, (Q^L_2)^+\}. \tag{2.4}$$

Hence from the assumption that a state with positive energy exists it follows that $c^{LL} > 0$. Therefore we can normalise Q^L_α so that $c^{LM} = \delta^{LM}$.

Consider now $[Q^L_\alpha, P_{\beta\dot\gamma}]$. The symmetric part in the indices α, β belongs to the representation $(1, \frac{1}{2})$. Since there is no such thing it must vanish. The antisymmetric part transforms in the same way as the $\bar{Q}^M_{\dot\gamma}$. Hence we have

$$[Q^L_\alpha, P_{\beta\dot\gamma}] = \epsilon_{\alpha\beta} b^{LM} \bar{Q}^M_{\dot\gamma}. \tag{2.5}$$

Next, we have to find out what $\{Q^L_\alpha, Q^M_\beta\}$ is. The antisymmetric part in the indices α, β is a scalar and hence a linear combination of the B_l. The symmetric part transforms in the same way as $M_{\alpha\beta}$. Therefore we have

$$\{Q^L_\alpha, Q^M_\beta\} = a^{LM} M_{\alpha\beta} + \epsilon_{\alpha\beta} B^{LM} \tag{2.6}$$

where

$$B^{LM} = \sum_l (a^l)^{LM} B_l. \tag{2.7}$$

We now determine the coefficients a^{LM} and b^{LM} with the help of the Jacobi identity

$$[O_1, \{O_2, O_3\}] + \{O_2, [O_3, O_1]\} - \{O_3, [O_1, O_2]\} = 0 \tag{2.8}$$

valid for arbitrary operators. The $(P_{\delta\dot\gamma}, Q^L_\alpha, Q^M_\beta)$ identity yields

$$\epsilon_{\beta\delta}b^{ML}P_{\alpha\dot\gamma}+\epsilon_{\alpha\delta}b^{LM}P_{\beta\dot\gamma}=\mathrm{i}\alpha^{LM}(\epsilon_{\delta\alpha}P_{\beta\dot\gamma}+\epsilon_{\delta\beta}P_{\alpha\dot\gamma}). \tag{2.9}$$

This implies

$$a^{LM}=\mathrm{i}b^{LM} \tag{2.10}$$

$$b^{LM}=b^{ML} \tag{2.11}$$

From the $(P_{\delta\dot\gamma}, Q^L_\alpha, \bar Q^M_{\dot\beta})$ identity we obtain

$$\begin{aligned}\mathrm{i}\rho\delta^{LM}&(\epsilon_{\delta\alpha}\bar M_{\dot\gamma\dot\beta}+\epsilon_{\dot\gamma\dot\beta}M_{\delta\alpha})\\ &=-\epsilon_{\dot\beta\dot\gamma}\overline{b^{MM'}}a^{LM'}M_{\alpha\delta}+\epsilon_{\alpha\delta}b^{LM'}\overline{\alpha^{MM'}}\bar M_{\dot\beta\dot\gamma}\\ &\quad-\epsilon_{\dot\beta\dot\gamma}\overline{b^{MM'}}\epsilon_{\alpha\delta}B^{LM'}+\epsilon_{\alpha\delta}b^{LM'}\epsilon_{\dot\beta\dot\gamma}(B^{MM'})^{+}.\end{aligned} \tag{2.12}$$

This implies

$$(bb^{+})^{LM}=\rho\delta^{LM} \tag{2.13}$$

$$b^{+M'M}B^{LM'}=b^{LM'}(B^{MM'})^{+}. \tag{2.14}$$

Equation (2.13) implies $\rho>0$. It implies further that $d=\rho^{-1/2}b$ is unitary. Because of (2.11) d is symmetric. Therefore d can be written in the form $d=\exp(if)$ where f is real and symmetric (Gantmacher 1958). Hence there is a real orthogonal matrix g such that gfg^{-1} is diagonal. Accordingly, we have

$$gdg^{-1}=\exp(\mathrm{i}gfg^{-1})=\begin{pmatrix}\exp(\mathrm{i}\delta_1) & & 0\\ & \exp(\mathrm{i}\delta_2) & \\ 0 & & \ddots\end{pmatrix}$$

with real δ_i. If we define new Q^L_α as

$$\exp(-\mathrm{i}\delta_L/2)g^{LL'}Q^{L'}_\alpha$$

then these new Q^L_α satisfy (1.9)–(1.11).

Equation (2.14) implies (1.12). There is another identity valid for arbitrary operators

$$[O_1,\{O_2,O_3\}]+[O_2,\{O_3,O_1\}]+[O_3,\{O_1,O_2\}]=0. \tag{2.15}$$

Applying this to $\bar Q^N_{\dot\gamma}$, Q^L_α and Q^M_β, one obtains (1.13). Applying then (2.8) to A^{LM}, $Q^{L'}_\alpha$ and $Q^{M'}_\beta$, one obtains (1.14). We shall now show that the A^{MN} with $N>M$ are linearly independent. Suppose

$$\sum_{N>M}a_{MN}A^{MN}=0.$$

If we commute this equation with Q^L_α we obtain

$$\sum_{N=1}^{L-1}a_{NL}Q^N_\alpha-\sum_{N=L+1}^{\nu}\alpha_{LN}Q^N_\alpha=0.$$

Since the Q^N_α are linearly independent it follows that $a_{MN}=0$.

Consider now the commutator $[Q^L_\alpha, A]$ with $A\in I$. Since it transforms under Lorentz transformations in the same way as the Q^L_α we have

$$[Q^L_\alpha, A]=s^{LM}(A)Q^M_\alpha. \tag{2.16}$$

One can check easily that the matrices s form a representation of the Lie algebra I. Applying then (2.8) to $A \in I$, Q_α^L, and Q_β^M one finds

$$\epsilon_{\alpha\beta}([A, A^{LM}] + s^{MM'}(A)A^{LM'} - s^{LM'}(A)A^{MM'}) \\ = -\mathrm{i}(s^{ML}(A) + s^{LM}(A))M_{\alpha\beta}. \quad (2.17)$$

This implies

$$s^{ML}(A) = -s^{LM}(A), \quad (2.18)$$

$$[A, A^{LM}] = s^{LM'}(A)A^{MM'} - s^{MM'}(A)A^{LM'}. \quad (2.19)$$

From (2.19) and (1.14) it follows that I_1 is an invariant sub-algebra. K is the kernel of the representation s and hence is also an invariant sub-algebra. If

$$A = \sum_{N>M} a_{MN}A^{MN} \quad (2.20)$$

then it follows from (1.13) that

$$s^{LM}(A) = a_{LM} \qquad \text{for } M > L. \quad (2.21)$$

This implies that the restriction s_{I_1} of s to I_1 is a one-to-one mapping of I_1 onto $s(I)$. Let i be the restriction of the canonical map $A \in I \to A + K \in I/K$ to I_1 and let s' be the one-to-one mapping of I/K into $s(I)$ induced by s. Then i is injective because $s'i = s_{I_1}$ is injective and i is surjective because $(s')^{-1} = i(s_{I_1})^{-1}$ is surjective. Hence i is a one-to-one mapping of I_1 onto I/K. Therefore $I = I_1 \oplus K$.

References

Das A 1977 *Phys. Rev.* D **15** 2805
Deser S and Zumino B 1977 *Phys. Rev. Lett.* **38** 1433
Ferrara S 1977 *Phys. Lett.* **69B** 481
Ferrara S, Scherk J and Zumino B 1977 *Phys. Lett.* B **66** 35
Ferrara S and Van Nieuwenhuizen P 1976 *Phys. Rev. Lett.* **37** 1669
Freedman D Z 1977 *Phys. Rev. Lett.* **38** 105
Freedman D Z and Das A 1977 *Nucl. Phys.* B **120** 221
Freund P G O and Kaplansky I 1976 *J. Math. Phys.* **17** 228
Gantmacher F R *Matrizenrechnung* II 1958 (Berlin: VEB Deutscher Verlag der Wissenschaften) p 4
Haag R, Łopuszański J T and Sohnius M *Nucl. Phys.* B **88** 257
Nahm W 1977 *Nucl. Phys.* B **135** 149
de Wit B and Freedman D Z 1977 *Nucl. Phys.* B **130** 105